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Integrable stochastic ladder models

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Abstract
A general way to construct ladder models with certain Lie algebraic or quantum Lie algebraic symmetries is presented. These symmetric models give rise to a series of integrable systems. It is shown that corresponding to these $SU(2)$ symmetric integrable ladder models there are exactly solvable stationary discrete-time (resp. continuous-time) Markov processes with transition matrices (resp. intensity matrices) having spectra which coincide with the ones of the corresponding integrable models.

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Integrable models have played significant roles in statistical and condensed matter physics. Some of them have been obtained and investigated using an algebraic or coordinate 'Bethe ansatz method' [1, 2]. The intrinsic symmetry of these integrable chain models plays an essential role in finding complete sets of eigenstates of the systems. On the other hand, stochastic models like stochastic reaction–diffusion models, models describing coagulation/decoagulation, birth/death processes, pair-creation/pair-annihilation of molecules on a chain, have attracted considerable interest due to their importance in many physical, chemical and biological processes [3][4]. The theoretical description of stochastic reaction–diffusion systems is given by the 'master equation' which describes the time evolution of the probability distribution function [4, 5]. This equation has the form of a heat equation with potential (i.e. a Schrödinger equation with 'imaginary time'). If an integrable system with an open boundary condition can be transformed into a stochastic reaction–diffusion system, e.g. by a unitary transformation between their respective Hamiltonians, looked upon as self-adjoint operators acting in the respective Hilbert spaces, then the stochastic model so obtained is exactly solvable with the same energy spectrum as the one of the integrable system [4, 6, 7].

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[4] For a review and an extensive literature list concerning physical aspects of the models, see e.g. [3].

In [8], we have presented a general procedure to construct open chain models having a certain Lie algebra or quantum Lie algebra symmetry by using the coproduct properties of bi-algebras. These models can be reduced to integrable ones via a detailed representation of the symmetry algebras involved. In recent years spin ladders have attracted considerable attention, due to the developing experimental results on ladder materials and the hope to get some insight into the physics of metal-oxide superconductors [9]. In this Letter we study the construction of ladder models with certain Lie algebra or quantum Lie algebra symmetry. We show that the integrable quantum spin ladder model discussed in [10] can be obtained in this way and it can be transformed into both stationary discrete-time (discrete reaction–diffusion models) and stationary continuous-time Markov processes with transition matrices resp. intensity matrices having the same spectra as the ones of this $SU(2)$ invariant integrable ladder model.

Let A be an associative Lie bi-algebra with basis $e = \{e_\alpha\}$, $\alpha = 1, 2, \ldots, n$, satisfying the Lie commutation relations $[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma$, where $C^\gamma_{\alpha\beta}$ are the structure constants with respect to the base e. Let Δ (resp. $C(e)$) be the coproduct operator (resp. Casimir operator) of the algebra A. The coproduct operator action on the Lie algebra elements is given by $\Delta e_\alpha = e_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes e_\alpha$, $\mathbf{1}$ stands for the identity operator. It can be immediately checked that $[\Delta e_\alpha, \Delta e_\beta] = C^\gamma_{\alpha\beta} \Delta e_\gamma$ and $[\Delta C(e), \Delta e_\alpha] = 0$, $\alpha = 1, 2, \ldots, n$.

Let us consider a two-leg ladder with L rungs. To each point at the ith rung, $i = 1, \ldots, L$, and θth leg, $\theta = 1, 2$, of the ladder we associate a (finite-dimensional complex) Hilbert space H_i^θ. We can then associate to the whole ladder the tensor product $H_1^1 \otimes H_1^2 \otimes H_2^1 \otimes H_2^2 \otimes \cdots \otimes H_L^1 \otimes H_L^2$. The generators of the algebra A acting on this Hilbert space associated with the above ladder are given by $E_\alpha = \Delta^{2L-1} e_\alpha$, $\alpha = 1, 2, \ldots, n$, where we have defined

$$\Delta^m = (\underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{m \text{ times}} \otimes \Delta) \ldots (\mathbf{1} \otimes \mathbf{1} \otimes \Delta)(\mathbf{1} \otimes \Delta)\Delta \qquad \forall m \in \mathbb{N}. \tag{1}$$

E_α also generates the Lie algebra A: $[E_\alpha, E_\beta] = C^\gamma_{\alpha\beta} E_\gamma$.

Let

$$h = \sum_{i=1}^{3} \sum_{j=1}^{2} a_{ij} \Delta_i^2 \Delta_j^1 \Delta C(e) \tag{2}$$

where $\Delta_1^1 = (\Delta \otimes \mathbf{1})$, $\Delta_2^1 = (\mathbf{1} \otimes \Delta)$, $\Delta_1^2 = (\Delta \otimes \mathbf{1} \otimes \mathbf{1})$, $\Delta_2^2 = (\mathbf{1} \otimes \Delta \otimes \mathbf{1})$, $\Delta_3^2 = (\mathbf{1} \otimes \mathbf{1} \otimes \Delta)$, $a_{ij} \in \mathbb{C}$ such that h is Hermitian. Let $\mathbb{F}$ denote a real entire function defined on the $2L$th tensor space $A \otimes A \otimes \cdots \otimes A$ of the algebra A. We call

$$H = \sum_{i=1}^{L-1} \mathbb{F}(h)_{i,i+1} \tag{3}$$

the (quantum mechanics) Hamiltonian associated with the ladder. Here $\mathbb{F}(h)_{i,i+1}$ means that the four-fold tensor element $\mathbb{F}(h)$ is associated with the i and $(i+1)$th rungs of the ladder and acts on the space $H_i^1 \otimes H_i^2 \otimes H_{i+1}^1 \otimes H_{i+1}^2$, i.e.

$$\mathbb{F}(h)_{i,i+1} = \mathbf{1}_1^1 \otimes \mathbf{1}_1^2 \otimes \cdots \otimes \mathbf{1}_{i-1}^1 \otimes \mathbf{1}_{i-1}^2 \otimes \mathbb{F}(h) \otimes \mathbf{1}_{i+2}^1 \otimes \mathbf{1}_{i+2}^2 \otimes \cdots \otimes \mathbf{1}_L^1 \otimes \mathbf{1}_L^2. \tag{4}$$

Theorem 1. *The Hamiltonian H is a self-adjoint operator acting in $H_1^1 \otimes H_1^2 \otimes H_2^1 \otimes H_2^2 \otimes \cdots \otimes H_L^1 \otimes H_L^2$ and is invariant under the algebra A.*

Proof. That H is self-adjoint is immediate from the definition. To prove the invariance of H it suffices to prove $[H, E_\alpha] = 0$, $\alpha = 1, 2, \ldots, n$.

From the formula for the above coproduct we have

$$E_\alpha = \sum_{i=1}^{L} (e_\alpha)_i \tag{5}$$

where $(e_\alpha)_i = \mathbf{1}_1^1 \otimes \mathbf{1}_1^2 \otimes \cdots \otimes \mathbf{1}_{i-1}^1 \otimes \mathbf{1}_{i-1}^2 \otimes (e_\alpha \otimes \mathbf{1}_i^2 \otimes + \mathbf{1}_i^1 \otimes e_\alpha) \otimes \mathbf{1}_{i+1}^1 \otimes \mathbf{1}_{i+1}^2 \otimes \cdots \otimes \mathbf{1}_L^1 \otimes \mathbf{1}_L^2$.

From $[\Delta C(e), \Delta e_\alpha] = 0$, $\alpha = 1, 2, \ldots, n$, it follows easily that $[h, \Delta^2 e_\alpha] = 0$, where $\Delta^2 e_\alpha = (\mathbf{1} \otimes \mathbf{1} \otimes \Delta)(\mathbf{1} \otimes \Delta)(\Delta) e_\alpha$, as defined in (1). Obviously $[\mathbb{F}(h)_{i,i+1}, (e_\alpha)_j] = 0$, $\forall j \neq i, i+1$. Therefore we have, for all $\alpha = 1, 2, \ldots, n$:

$$\begin{aligned}
[H, E_\alpha] &= \left[\sum_{i=1}^{L-1} \mathbb{F}(h)_{i,i+1}, \sum_{j=1}^{L-1} (e_\alpha)_j \right] \\
&= \sum_{i=1}^{L-1} \left[\mathbb{F}(h)_{i,i+1}, \sum_{j=1}^{i-1} (e_\alpha)_j + \sum_{k=i+2}^{L} (e_\alpha)_k + (e_\alpha)_i + (e_\alpha)_{i+1} \right] \\
&= \sum_{i=1}^{L-1} \left[\mathbb{F}(h)_{i,i+1}, (e_\alpha)_i + (e_\alpha)_{i+1} \right] \\
&= \sum_{i=1}^{L-1} \left[\mathbb{F}(h)_{i,i+1}, (\Delta^2 e_\alpha)_{i,i+1} \right] = 0.
\end{aligned} \tag{6}$$

□

Let V be a complex vector space and $\check{R}$ be the solution of the quantum Yang–Baxter equation (QYBE) [2, 11] without spectral parameters, see e.g. [12]. Then $\check{R}$ takes values in $\mathrm{End}_{\mathbb{C}}(V \otimes V)$. The QYBE is

$$\check{R}_{12} \check{R}_{23} \check{R}_{12} = \check{R}_{23} \check{R}_{12} \check{R}_{23} \tag{7}$$

where $\check{R}_{12} = \check{R} \otimes \mathrm{id}$, $\check{R}_{23} = \mathrm{id} \otimes \check{R}$ and id is the identity operator on V.

In the following we say that a ladder model having a (quantum mechanical) Hamiltonian of the form

$$H = \sum_{i=1}^{L-1} (\mathcal{H})_{i,i+1} \tag{8}$$

is integrable in the sense that the operator $\mathcal{H}$ satisfies the QYBE relation (7), i.e.

$$(\mathcal{H})_{12} (\mathcal{H})_{23} (\mathcal{H})_{12} = (\mathcal{H})_{23} (\mathcal{H})_{12} (\mathcal{H})_{23} \tag{9}$$

where $(\mathcal{H})_{12} = \mathcal{H} \otimes \mathrm{id}$ and $(\mathcal{H})_{23} = \mathrm{id} \otimes \mathcal{H}$. $\mathcal{H}$ is a solution of the Yang–Baxter equation without spectral parameters. Correspondingly the ith complex vector space V_i now stands for $H_i^1 \otimes H_i^2$. After Baxterization the Hamiltonian system (8) satisfying relation (9) can in principle be exactly solved by the algebraic Bethe ansatz method, see e.g. [1].

We consider ladder models with $SU(2)$ symmetry. Let S_i, $i = 1, 2, 3$, and C be the generators of the algebra $SU(2)$ and Casimir operator, respectively. The coproduct of the algebra is given by $\Delta S_i = \mathbf{1} \otimes S_i + S_i \otimes \mathbf{1}$, $i = 1, 2, 3$. Taking into account that $\Delta_i^j \mathbb{F}(e) = \mathbb{F}(\Delta_i^j e)$, $i = 1, 2, 3$, $j = 1, 2$, $\forall e \in SU(2)$, the generic h is of the form $\mathbb{F}(C_1, C_2, C_3)$, where

$$C_1 = \sum_{i=1}^{3} (S_i \otimes \mathbf{1} \otimes \mathbf{1} \otimes S_i + \mathbf{1} \otimes S_i \otimes \mathbf{1} \otimes S_i + \mathbf{1} \otimes \mathbf{1} \otimes S_i \otimes S_i)$$

$$C_2 = \sum_{i=1}^{3} (S_i \otimes \mathbf{1} \otimes \mathbf{1} \otimes S_i + S_i \otimes S_i \otimes \mathbf{1} \otimes \mathbf{1} + S_i \otimes \mathbf{1} \otimes S_i \otimes \mathbf{1})$$

$$C_3 = \sum_{i=1}^{3} (S_i \otimes \mathbf{1} \otimes \mathbf{1} \otimes S_i + \mathbf{1} \otimes S_i \otimes \mathbf{1} \otimes S_i + S_i \otimes \mathbf{1} \otimes S_i \otimes \mathbf{1} + \mathbf{1} \otimes S_i \otimes S_i \otimes \mathbf{1}).$$

In the spin-$\frac{1}{2}$ representation of the algebra $SU(2)$, the solutions of the QYBE (9) are 16×16 matrices. For instance, it is easy to check that

$$\begin{aligned}\mathcal{H}_0 = {} & \frac{108d - 55f}{108} C_{111} + \frac{-72d + 104f}{288} C_{112} + \frac{-486d + 211f}{270} C_{113} + \frac{-756d + 370f}{216} C_{121} \\ & - \frac{29f}{108} C_{122} + \frac{90d - 31f}{36} C_{123} + \frac{2d - f}{2} C_{131} + \frac{-54d + 26f}{108} C_{132} \\ & + \frac{-108d + 43f}{540} C_{133} + \frac{-216d + 80f}{864} C_{211} + \frac{11f}{108} C_{212} + \frac{216d - 119f}{108} C_{213}\end{aligned} \tag{10}$$

satisfies (9) for all $d, f \in \mathbb{R}$, where $C_{ijk} \equiv C_i \cdot C_j \cdot C_k$ and $i, j, k = 1, 2, 3$.

The corresponding solution related to the $SU(2)$-symmetric integrable ladder model in [10] can be expressed as

$$\begin{aligned}\mathcal{H} = {} & -\tfrac{5}{48} C_{111} - \tfrac{11}{32} C_{112} - \tfrac{61}{30} C_{113} - \tfrac{41}{48} (C_{121} - C_{122}) + \tfrac{21}{16} C_{123} \\ & + \tfrac{3}{4} C_{131} - \tfrac{17}{12} C_{132} + \tfrac{173}{240} C_{133} + \tfrac{55}{96} C_{211} - \tfrac{5}{3} C_{212} + \tfrac{131}{48} C_{213}.\end{aligned} \tag{11}$$

Through baxterization, $\mathcal{H}(x) = (x - 1)\mathcal{H} + 16 I_{16\times 16}$ satisfies the QYBE with spectral parameters: $\mathcal{H}_{12}(x)\mathcal{H}_{23}(xy)\mathcal{H}_{12}(y) = \mathcal{H}_{23}(y)\mathcal{H}_{12}(xy)\mathcal{H}_{23}(x)$, where $\mathcal{H}_{12}(\cdot) = \mathcal{H}(\cdot) \otimes I_{4\times 4}$, $\mathcal{H}_{23}(\cdot) = I_{4\times 4} \otimes \mathcal{H}(\cdot)$, $I_{n\times n}$ denotes the $n \times n$ identity matrix. The model can be exactly solved using a algebraic Bethe ansatz method. It describes a periodic spin ladder system with both isotropic exchange interactions and biquadratic interactions:

$$\begin{aligned}H = {} & \tfrac{1}{2} \sum_{i=1}^{L-1} (\tfrac{1}{2} + 2\boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{1,i+1})(\tfrac{1}{2} + 2\boldsymbol{S}_{2,i} \cdot \boldsymbol{S}_{2,i+1}) - \tfrac{1}{2} \sum_{i=1}^{L-1} (\tfrac{1}{2} + 2\boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{2,i+1})(\tfrac{1}{2} + 2\boldsymbol{S}_{2,i} \cdot \boldsymbol{S}_{1,i+1}) \\ & + \tfrac{5}{6} \sum_{i=1}^{L-1} (\tfrac{1}{2} + 2\boldsymbol{S}_{1,i} \cdot \boldsymbol{S}_{2,i})(\tfrac{1}{2} + 2\boldsymbol{S}_{1,i+1} \cdot \boldsymbol{S}_{2,i+1})\end{aligned}$$

where $\boldsymbol{S}_{\theta,i} = (\sigma^x_{\theta,i}, \sigma^y_{\theta,i}, \sigma^z_{\theta,i})/2$, $\sigma^x, \sigma^y, \sigma^z$ are Pauli matrices. $\boldsymbol{S}_{1,i}$ (resp. $\boldsymbol{S}_{2,i}$) is the spin operator on the first (resp. second) leg of the ith rung of the ladder.

It is further shown that for a more general form of (11),

$$\begin{aligned}\mathcal{H}' = {} & \frac{-45 + 23a - 4b - 28c}{432} C_{111} + \frac{-99 - 3a - 3b - c}{288} C_{112} \\ & + \frac{-1098 - 91a - 118b - 16c}{540} C_{113} + \frac{-369 - 97a - 70b + 50c}{432} C_{121} \\ & + \frac{396 + 4a + 31b + 25c}{432} C_{122} + \frac{189 + 29a + 20b - 4c}{144} C_{123} + \frac{3}{4} C_{131} \\ & + \frac{-306 - 2a - 29b - 14c}{216} C_{132} + \frac{1557 - 71a + 172b + 124c}{2160} C_{133} \\ & + \frac{495 - a + 53b + 47c}{864} C_{211} + \frac{-720 - 22a - 49b - 43c}{432} C_{212} \\ & + \frac{1179 + 91a + 118b + 16c}{432} C_{213}\end{aligned} \tag{12}$$

with $a, b, c \in \mathbb{R}$, the corresponding ladder model $H' = \sum_{i=1}^{L-1} \mathcal{H}'_{i,i+1}$ can also be exactly solved by a coordinate Bethe ansatz [10].

We now consider stochastic processes [13] on a ladder. Let (Ω, P) be a probability space, with Ω the finite sample space and P the probability measure defined on the σ-algebra of all subsets of Ω. For a discrete time stationary Markov chain $\{X_i\}$, $i \in \mathbb{N}$, with underlying probability space (Ω, P) and a finite state space $S = \{1, 2, 3, \ldots, m\}$, there are m^2 transition probabilities $\{p_{\alpha\beta}\}$, $\alpha, \beta = 1, 2, \dots, m$. The stochastic transition matrix $P = (p_{\alpha\beta})$ has the following properties:

$$p_{\alpha\beta} \geqslant 0 \qquad \sum_{\alpha=1}^{m} p_{\alpha\beta} = 1 \qquad \alpha, \beta = 1, 2, \dots, m. \tag{13}$$

For a stationary continuous-time real-valued stochastic process, $\{X_t\}_{t\in\mathbb{R}_+}$ (on the probability space (Ω, P)), the transition semigroup $P(t) = P(X_{t=j}|X_0 = i)$ is generated by an intensity matrix $Q = (q_{\alpha\beta})$ with the properties

$$q_{\alpha\beta} \geqslant 0 \quad \alpha \neq \beta \qquad q_{\alpha\alpha} = -\sum_{\alpha\neq\beta} q_{\alpha\beta} \quad \alpha, \beta = 1, 2, \dots, m. \tag{14}$$

The transition matrix P (resp. intensity matrix Q) defines the stochastic processes on a ladder. In the following we call a ladder associated with the above stochastic processes, for instance, particles jumping randomly on the ladder, characterized by the matrices P and Q a Markov ladder, though geometrically it is equivalent to a chain with particular non-nearest neighbour interactions. If the eigenvalues and eigenstates of P resp. Q are known, then exact results concerning the stochastic processes, such as time-dependent averages and correlations, can be obtained. We say that a Markov ladder is integrable (resp. $SU(2)$-symmetric) if the eigenvalues and eigenstates of the related transition matrix P or intensity matrix Q can be exactly solved (resp. is $SU(2)$ invariant).

To every site on the ith rung and θth leg of the ladder we associate states described by the variable τ_i^j taking values 0 and 1 (conventionally a vacancy at the site is associated with the state 0 and an occupied state is associated with the state 1). The state space of this algebraic ladder is then finite and has a total of $m = 2^{2L}$ states.

Theorem 2. *The following matrix:*

$$P_{SU(2)} = \frac{1}{4(L-1)(18+4a+4b+c)} \sum_{i=1}^{L-1} \mathcal{H}''_{i,i+1} \tag{15}$$

defines a stationary discrete-time $SU(2)$-symmetric integrable Markov ladder for a, b, $c \geqslant 0$, $a + 2b - 16 \geqslant 0$. The operator $\mathcal{H}''$ is given by

$$\mathcal{H}'' = \begin{pmatrix}
a_1 & a_2 & a_2 & a_2 & a_3 & a_4 & a_4 & a_4 & a_3 & a_4 & a_4 & a_4 & a_3 & a_4 & a_4 & a_4 \\
a_2 & a_5 & a_6 & a_6 & a_7 & a_3 & a_8 & a_8 & a_8 & a_9 & a_4 & a_4 & a_8 & a_9 & a_4 & a_4 \\
a_2 & a_6 & a_5 & a_6 & a_8 & a_4 & a_9 & a_4 & a_7 & a_8 & a_3 & a_8 & a_8 & a_4 & a_9 & a_4 \\
a_2 & a_6 & a_6 & a_5 & a_8 & a_4 & a_4 & a_9 & a_8 & a_4 & a_4 & a_9 & a_7 & a_8 & a_8 & a_3 \\
a_3 & a_7 & a_8 & a_8 & a_5 & a_2 & a_6 & a_6 & a_9 & a_8 & a_4 & a_4 & a_9 & a_8 & a_4 & a_4 \\
a_4 & a_3 & a_4 & a_4 & a_2 & a_1 & a_2 & a_2 & a_4 & a_3 & a_4 & a_4 & a_4 & a_3 & a_4 & a_4 \\
a_4 & a_8 & a_9 & a_4 & a_6 & a_2 & a_5 & a_6 & a_8 & a_7 & a_3 & a_8 & a_4 & a_8 & a_9 & a_4 \\
a_4 & a_8 & a_4 & a_9 & a_6 & a_2 & a_6 & a_5 & a_4 & a_8 & a_4 & a_9 & a_8 & a_7 & a_8 & a_3 \\
a_3 & a_8 & a_7 & a_8 & a_9 & a_4 & a_8 & a_4 & a_5 & a_6 & a_2 & a_6 & a_9 & a_4 & a_8 & a_4 \\
a_4 & a_9 & a_8 & a_4 & a_8 & a_3 & a_7 & a_8 & a_6 & a_5 & a_2 & a_6 & a_4 & a_9 & a_8 & a_4 \\
a_4 & a_4 & a_3 & a_4 & a_4 & a_4 & a_3 & a_4 & a_2 & a_2 & a_1 & a_2 & a_4 & a_4 & a_3 & a_4 \\
a_4 & a_4 & a_8 & a_9 & a_4 & a_4 & a_8 & a_9 & a_6 & a_6 & a_2 & a_5 & a_8 & a_8 & a_7 & a_3 \\
a_3 & a_8 & a_8 & a_7 & a_9 & a_4 & a_4 & a_8 & a_9 & a_4 & a_4 & a_8 & a_5 & a_6 & a_6 & a_2 \\
a_4 & a_9 & a_4 & a_8 & a_8 & a_3 & a_8 & a_7 & a_4 & a_9 & a_4 & a_8 & a_6 & a_5 & a_6 & a_2 \\
a_4 & a_4 & a_9 & a_8 & a_4 & a_4 & a_9 & a_8 & a_8 & a_8 & a_3 & a_7 & a_6 & a_6 & a_5 & a_2 \\
a_4 & a_4 & a_4 & a_3 & a_4 & a_4 & a_4 & a_3 & a_4 & a_4 & a_4 & a_3 & a_2 & a_2 & a_2 & a_1
\end{pmatrix} \tag{16}$$

where $a_1 = 66+a+4b+4c$, $a_2 = -10+a+2b$, $a_3 = 6+a+2b$, $a_4 = 2+a$, $a_5 = 54+a+4b+4c$, $a_6 = -16+a+2b$, $a_7 = 14+a$, $a_8 = 8+a$, $a_9 = a+2b$. $\mathcal{H}''_{i,i+1}$ acts on the i and $i+1$ rungs as defined in (4).

Proof. For the integrable ladder model with Hamiltonian $H' = \sum_{i=1}^{L-1} \mathcal{H}'_{i,i+1}$, the system remains integrable if one adds to H' a constant term and multiplies H' by a constant factor. Moreover the spectrum of H' will not be changed if one changes the local basis of the rungs, i.e. the following Hamiltonian H'', defined by

$$H'' = BH'B^{-1} \qquad B = \otimes_{i=1}^{L} B_i \tag{17}$$

where B_i are 4×4 non-singular matrices, has the same eigenvalues as H'.

It is straightforward to prove that $\mathcal{H}'' = B\mathcal{H}'B^{-1}$, where

$$B = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 1/2 & -1/2 & 1 \\ 0 & -1/2 & -3/2 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Therefore the Hamiltonian systems H' and $H'' = \sum_{i=1}^{L-1} \mathcal{H}''_{i,i+1}$ satisfy the relation (17) with $B_i = B$, $i = 1, 2, \ldots, L$. Hence H'' is also $SU(2)$-symmetric and integrable with the same spectrum as H'.

For $a + 2b \geqslant 0$, as the entries of $\mathcal{H}''$ are positive, $H''_{\alpha\beta} \geqslant 0$, $\alpha, \beta = 1, 2, \ldots, 2^{2L}$. From (16) we also have $\sum_{\alpha=1}^{16} \mathcal{H}''_{\alpha\beta} = 4(18 + 4a + 4b + c)$, $\forall\, \beta = 1, 2, \ldots, 16$. By the definition (13) $P_{SU(2)}$ is the transition matrix of a stationary discrete-time $SU(2)$-symmetric integrable Markov ladder. □

The state space of this Markov processes associated with the stochastic matrix $P_{SU(2)}$ is $S = (1, 2, \ldots, 2^{2L})$. Generally there is no closed subset C of the state space S such that $(P_{SU(2)})_{ij} = 0$ for all $i \in C$ and $j \notin C$. In a certain parameter region of the a, b, c there are nonempty closed sets other than S itself and the Markov ladder becomes reducible. However, there exists no absorbing state in this Markov ladder.

By using results in the proof of theorem 2, we have also the following integrable stationary continuous-time Markov ladder:

Theorem 3. *The matrix*

$$Q_{SU(2)} = H'' - 4(L-1)(18 + 4a + 4b + c) = \sum_{i=1}^{L-1} (\mathcal{H}'' - 4(18 + 4a + 4b + c))_{i,i+1} \tag{18}$$

is the intensity matrix of a stationary continuous-time Markov ladder.

We have discussed the construction of integrable ladder models with Lie algebra symmetry. It is shown that the stochastic processes correspond to the $SU(2)$ symmetric integrable ladder models define exactly solvable stationary discrete-time (resp. continuous-time) Markov ladder with transition matrices (resp. intensity matrices) which coincide with those of the corresponding integrable models.

Integrable ladder models with quantum algebraic symmetry and the related Markov processes can be investigated in a similar way. Let $e = \{e_\alpha, f_\alpha, h_\alpha\}$, $\alpha = 1, 2, \ldots, n$, be the Chevalley basis of a Lie algebra A with rank n. Let $e' = \{e'_\alpha, f'_\alpha, h'_\alpha\}$, $\alpha = 1, 2, \ldots, n$, be the corresponding elements of the quantum (q-deformed) Lie algebra A_q. We denote by r_α the simple roots of the Lie algebra A. The Cartan matrix $(a_{\alpha\beta})$ is then

$$a_{\alpha\beta} = \frac{1}{d_\alpha}(r_\alpha \cdot r_\beta) \qquad d_\alpha = \tfrac{1}{2}(r_\alpha \cdot r_\alpha). \tag{19}$$

The coproduct operator Δ' of the quantum algebra A_q is given by

$$\Delta' h'_\alpha = h'_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes h'_\alpha \tag{20}$$

$$\Delta' e'_\alpha = e'_\alpha \otimes q^{-d_\alpha h'_\alpha} + q^{d_\alpha h'_\alpha} \otimes e'_\alpha \tag{21}$$

$$\Delta' f'_\alpha = f'_\alpha \otimes q^{-d_\alpha h'_\alpha} + q^{d_\alpha h'_\alpha} \otimes f'_\alpha \tag{22}$$

$q \in \mathbb{C}$, $q^{d_\alpha} \neq \pm 1, 0$. In the following we use the notations Δ'^m and Δ'^i_j defined similarly as in (1) and (2).

Theorem 4. *The ladder model defined by the following Hamiltonian acting in* $H_1^1 \otimes H_1^2 \otimes H_2^1 \otimes H_2^2 \otimes \cdots \otimes H_L^1 \otimes H_L^2$ *is invariant under the quantum algebra* A_q*:*

$$H_q = \sum_{i=1}^{L-1} \mathbb{F}(h_q)_{i,i+1} \tag{23}$$

where $h_q = \sum_{i=1}^3 \sum_{j=1}^2 a_{ij} \Delta_i'^2 \Delta_j'^1 \Delta C_q(e')$, $C_q(e')$ *is the Casimir operator of* A_q.

Proof. The generators of A_q on the ladder are given by

$$\begin{aligned} H'_\alpha &= \Delta'^{2L-1} h'_\alpha = \sum_{i=1}^{L} \mathbf{1}_1^1 \otimes \mathbf{1}_1^2 \otimes \cdots (h'_\alpha \otimes \mathbf{1}_i^2 \otimes + \mathbf{1}_i^1 \otimes h'_\alpha) \otimes \cdots \otimes \mathbf{1}_L^1 \otimes \mathbf{1}_L^2 \\ E'_\alpha &= \sum_{i=1}^{L} q^{d_\alpha h'_\alpha} \otimes \cdots \otimes (e'_\alpha \otimes \mathbf{1}_i^2 \otimes + \mathbf{1}_i^1 \otimes e'_\alpha) \otimes \cdots \otimes q^{-d_\alpha h'_\alpha} \\ F'_\alpha &= \sum_{i=1}^{L} q^{d_\alpha h'_\alpha} \otimes \cdots \otimes (f'_\alpha \otimes \mathbf{1}_i^2 \otimes + \mathbf{1}_i^1 \otimes f'_\alpha) \otimes \cdots \otimes q^{-d_\alpha h'_\alpha}. \end{aligned} \tag{24}$$

From $[\Delta'\mathbb{F}(C_q(e')), \Delta' a] = 0$, $\forall a \in A_q$ and $\Delta' q^{\pm d_\alpha h'_\alpha} = q^{\pm d_\alpha h'_\alpha} \otimes q^{\pm d_\alpha h'_\alpha}$, we have $[h_q, \Delta'^2 h'_\alpha] = [h_q, \Delta'^2 e'_\alpha] = [h_q, \Delta'^2 f'_\alpha] = 0$. Therefore

$$\begin{aligned} [H_q, E'_\alpha] = \sum_{i=1}^{L-1} \big[& \mathbb{F}(h_q)_{i,i+1}, (e'_\alpha)_i^1 \otimes (q^{-d_\alpha h'_\alpha})_i^2 \otimes (q^{-d_\alpha h'_\alpha})_{i+1}^1 \otimes (q^{-d_\alpha h'_\alpha})_{i+1}^2 \\ & + (q^{d_\alpha h'_\alpha})_i^1 \otimes (e'_\alpha)_i^2 \otimes (q^{-d_\alpha h'_\alpha})_{i+1}^1 \otimes (q^{-d_\alpha h'_\alpha})_{i+1}^2 \\ & + (q^{d_\alpha h'_\alpha})_i^1 \otimes (q^{d_\alpha h'_\alpha})_i^2 \otimes (e'_\alpha)_{i+1}^1 \otimes (q^{-d_\alpha h'_\alpha})_{i+1}^2 \\ & + (q^{d_\alpha h'_\alpha})_i^1 \otimes (q^{d_\alpha h'_\alpha})_i^2 \otimes (q^{d_\alpha h'_\alpha})_{i+1}^1 \otimes (e'_\alpha)_{i+1}^2 \big] \\ = \sum_{i=1}^{L-1} & \big[\mathbb{F}(h_q), \Delta'^2(e'_\alpha) \big]_{i,i+1} = 0. \end{aligned}$$

$[H_q, F'_\alpha] = 0$ is obtained similarly. $[H_q, H'_\alpha] = 0$ can be proved like (6). Hence H_q commutes with the generators of A_q for the ladder. □

The Hamiltonian system (23) is expressed by the quantum algebraic generators $e' = (h'_\alpha, e'_\alpha, f'_\alpha)$. Assume now that $e \to e'(e)$ is an algebraic map from A to A_q (we remark that for rank one algebras, both classical and quantum algebraic maps can be discussed in terms of the two-dimensional manifolds related to the algebras, see [14]). We then have

$$H_q = \sum_{i=1}^{L-1} \mathbb{F}(h_q(e'(e))_{i,i+1}. \tag{25}$$

In this way we obtain ladder models having quantum algebraic symmetry but expressed in terms of the usual Lie algebraic generators $\{e_\alpha\}$ with manifest physical meaning.

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